The Magic Number - What it means and tying it down with simulation.

If you have been in this business long enough you will have heard the term the magic number. The magic number refers to the percentage of front lateral load transfer distribution at which the car produces the most grip. Like with a lot of things in this business it tends to get thrown around like a bad football with some people swearing by it and others thinking it is totally irrelevant. However if you ever have to construct a simulation model you are in a very strong position to put some numbers to this discussion. This will be the focus of this article.

The focus of the article is how we can use a tyre model derived from actual data to delve into what the magic number actually means. What your about to see is a fantastic case study of one of the by products of race car simulation. One of my perpetual frustrations with this business is how many engineers truly have no idea of what they have in their hands when they use a racecar simulation package. I have said time and time again the lap time and data you get is the full stop at the end of the process. As your about to see what you learn in the process is the pay off and what we are about to discuss is a perfect case in point.

To kick of this discussion let's return to the most basic tyre model you can have which is the second order traction circle radius vs load curve fit for a tyre. Mathematically this is what we are talking about,

$$TC_{RAD} = k_a(1 - k_b \cdot F_z) \cdot F_z \qquad (1)$$

where

TC_{rad} = Traction Circle radius (N)

k_a = initial coefficient of friction

k_b = drop off of coefficient with load

F_z = load on the tyre (N)

Some typical values for this is presented below,

Table 1 – Typical open wheeler numbers for maximum tyre force with the coefficient of friction dropping off linearly with load.

Parameter	Value
k_a	2
k_b	5.0 e-5 (1/N)

When you plot this out you'll have something that looks like Fig 1,

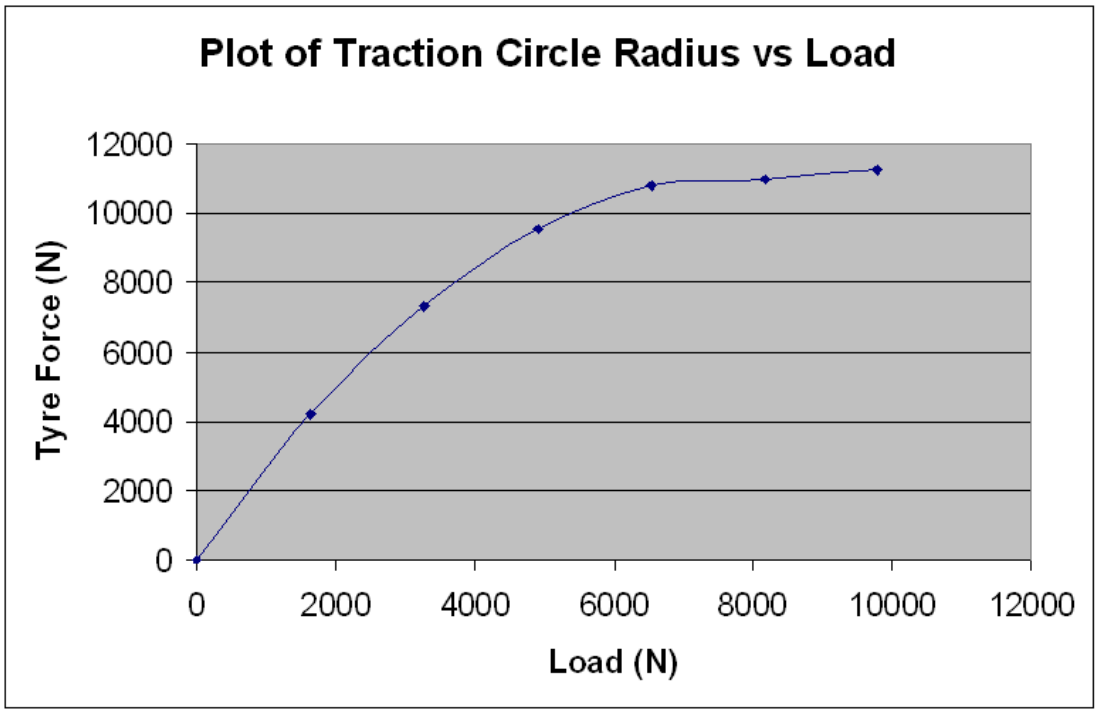

Fig - 1 Second order plot of the traction circle vs load characteristic.

As we discussed in detail in one of my most recent articles, tyre modelling from scratch, using a simulator such as ChassisSim you can fill in the blanks for this very quickly.

However where things get really interesting is when we take into account lateral load transfer and use equation (1) to quantify what will happen to the lateral forces on the tyres. For a given load transfer couple we have,

$$\begin{aligned} L_1 &= L_0 + \Delta L \\ L_2 &= L_0 - \Delta L \\ F_y &= k_a \cdot \left(\left(1 - k_b L_1\right) \cdot L_1 + \left(1 - k_b L_2\right) \cdot L_2 \right) \\ &= 2 \cdot k_a \cdot \left(1 - k_b L_0\right) \cdot L_0 - 2 \cdot k_a \cdot k_b \cdot \Delta L^2 \\ &= 2 \cdot TC_{RAD}\left(L_0\right) - 2 \cdot k_a \cdot k_b \cdot \Delta L^2 \end{aligned} \qquad (2)$$

Given a lateral load transfer factor of pr the front and rear load deltas will be given by,

$$\begin{aligned} \Delta L_F &= \frac{pr \cdot m_t \cdot a_y \cdot h}{tm} \\ \Delta L_R &= \frac{\left(1 - pr\right) \cdot m_t \cdot a_y \cdot h}{tm} \end{aligned} \qquad (3)$$

Here we have

ΔL_F = Delta load at the front (N)
ΔL_R = Delta load at the rear (N)

m_t = Total mass (kg)
pr = lateral load transfer (scaled from 0 - 1)
a_y = Lateral acceleration (m/s^2)
h = centre of gravity height (m)
tm = Mean track (m)

At this point in the discussion you may be thinking so what? Well what all this mathematical gobbledegook actually means is that for a given lateral load transfer distribution, a given lateral acceleration and a given tyre model we can calculate the maximum possible grip for a given cornering situation. Mathematically this can be expressed as,

$$\begin{aligned} F_{yt} &= 2 \cdot TC_{RAD}(L_{SF}) + 2 \cdot TC_{RAD}(L_{SR}) \\ &- 2 \cdot k_{af} \cdot k_{bf} \cdot \left(\frac{pr \cdot m_t \cdot a_y \cdot h}{tm} \right)^2 \\ &- 2 \cdot k_{ar} \cdot k_{br} \cdot \left(\frac{(1 - pr) \cdot m_t \cdot a_y \cdot h}{tm} \right)^2 \end{aligned} \tag{4}$$

Here we have,
F_{yt} = Total lateral force in N
L_{SF} = Front corner weight in N
L_{SR} = Rear corner weight in N
k_{af} = Front tyre initial coefficient of friction
k_{bf} = Front tyre drop off of coefficient with load
k_{ar} = Rear tyre initial coefficient of friction
k_{br} = Rear tyre drop off of coefficient with load

Where things get very interesting is when we expand equation (4) because it becomes a function of the lateral load transfer distribution. Expanding equation (4) we get,

$$\begin{aligned} F_{yt} &= F_{y0} - pr^2 \cdot (2 \cdot k_{af} \cdot k_{bg} + 2 \cdot k_{ar} \cdot k_{br}) \cdot \left(\frac{m_t \cdot a_y \cdot h}{tm} \right)^2 \\ &+ 4 \cdot pr \cdot k_{ar} \cdot k_{br} \cdot \left(\frac{m_t \cdot a_y \cdot h}{tm} \right)^2 \end{aligned} \tag{5}$$

In this case the F_{y0} term is all the non lateral load transfer terms in N. Where things get really interesting is when we derive equation (5) by the lateral load transfer term pr. Setting this to 0 the lateral load transfer distribution that will give us the most grip is given by,

$$pr = \frac{k_{ar} \cdot k_{br}}{k_{af} \cdot k_{bf} + k_{ar} \cdot k_{br}} \tag{6}$$

This is the origin of the magic number and if you have done the home work using your simulation software to reverse engineer the tyres from race data you can plot this out. This is illustrated in Fig-2,

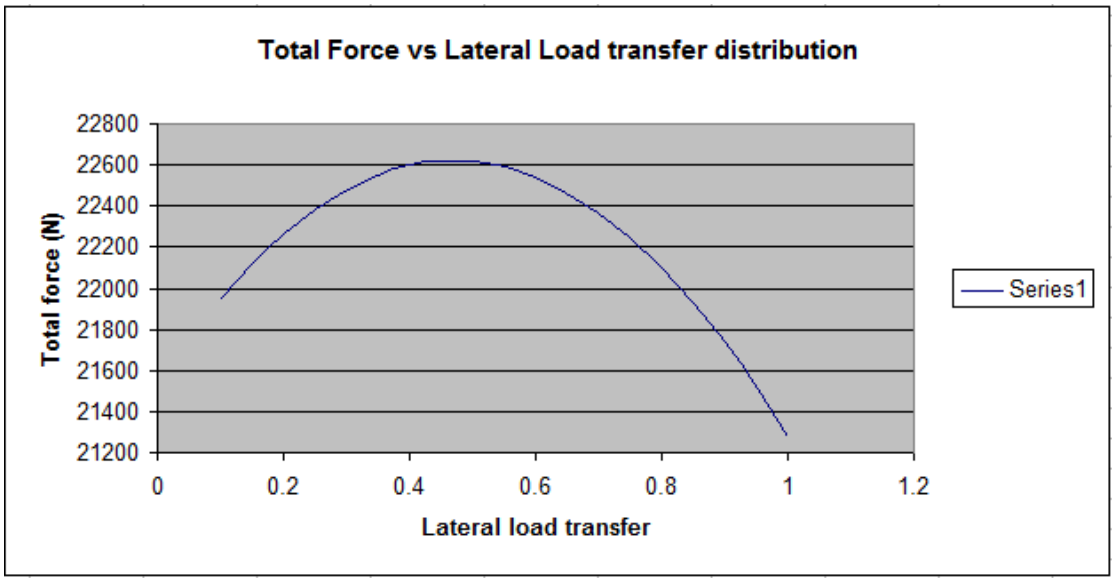

Fig-2 - A plot of maximum lateral grip vs lateral load transfer distribution

What is even more exciting is that it will tell you the sensitivities of this which in the above case was 1400 N. What equation (6) tells us is the magic number is purely a function of the tyres and what they want.

However there is more to this story than meets the eye. In equation (6) we have considered the tyres purely in cornering. Let's now consider the case where we have identical tyres front and rear but the car is Rear wheel drive. Let's say we multiply the available force from the rear tyres by a factor of e_{TE} which is scaled between 0 and 1. The case of 1 represent we have all of the available traction circle radius for cornering and 0 is when it is all used for accelerating. Equation (6) will become the following,

$$pr = \frac{e_{TE} \cdot k_{ar} \cdot k_{br}}{k_{af} \cdot k_{bf} + e_{TE} \cdot k_{ar} \cdot k_{br}} \qquad (7)$$

Let's put some numbers to this. Let's set e_{TE} to 0.9 and assume the tyres are the same front to rear. Since all the k_a and k_b terms are identical we have,

$$pr = \frac{0.9}{1+0.9} = 0.473$$

What this is saying is that for this Rear wheel drive case we want 47.3% of the load transfer at the front. So for maximum lateral grip we want the car to have it's weight transfer biased to the rear. All of us who have ever run a rear wheel drive car know that this is sure fire recipe for disaster.

What this case study illustrates is that the maximum lateral grip will have the potential to push us into unstable territory. Before you all start hitting the rev limiter just

remember one of the biggest advances in fighter aircraft design was when fighter aircraft designers recognised the performance potential in making their aircraft unstable. This trend was kicked off by the F-16 and has come to full maturity in the extreme plus agility designs you see with the Russian Sukhoi Su-35S and the Su-50 PAK-FA. They are unstable because that is where the performance is and it is no different then what we have seen with the magic number.

What will result in the final selection of the magic number is choosing a value that not only gives us good grip but also satisfies car stability. As a case in point let's revisit our rear wheel drive scenario but let's plot out the lateral grip with an ellipse factor of 0.9. This is illustrated in Fig-3,

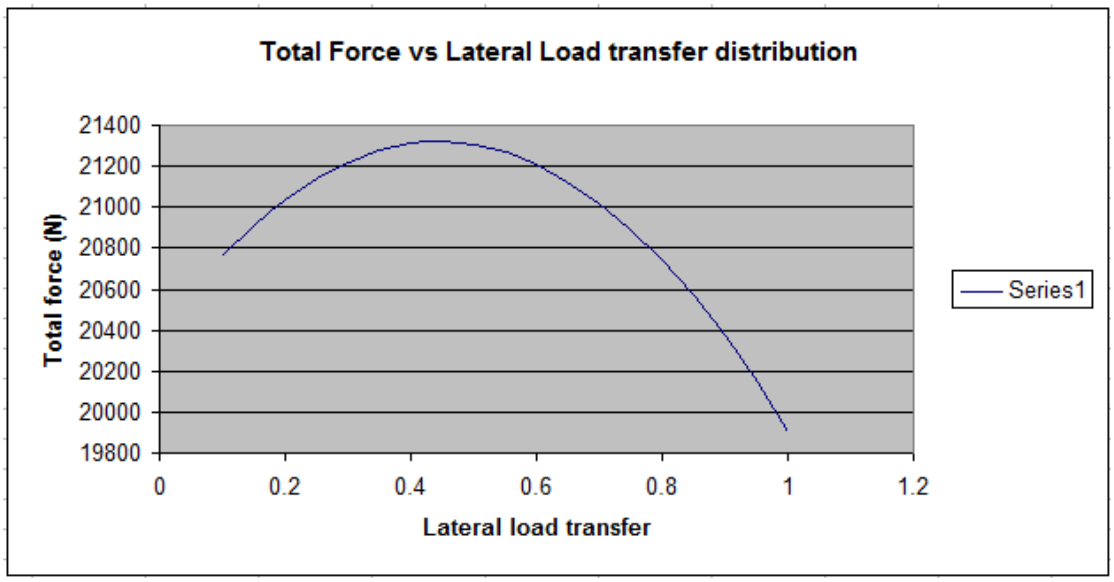

Fig-3 - Total force vs Lateral Load transfer with RWD taken into account

Take a look in the neighbourhood of lateral grip between 0.4 - 0.5. The lateral grip hardly drops off. What this means is that we have plenty of room to tune for race car stability to ensure that not only we have grip, but it is useable grip.

To quantify this we will use the stability index. To refresh everyone's memory the stability index is calculated by,

$$
\begin{aligned}
C_f &= \frac{\partial C_f}{\partial \alpha_f}\bigg|_{\alpha=\alpha_f} \cdot (F_{m1} + F_{m2}) \\
C_r &= \frac{\partial C_r}{\partial \alpha_r}\bigg|_{\alpha=\alpha_r} \cdot (F_{m3} + F_{m4}) \\
C_T &= C_f + C_r \\
stbi &\approx \frac{a \cdot C_f - b \cdot C_r}{C_T \cdot wb}
\end{aligned}
\qquad (8)
$$

Here we have,

$dC_F/da(\alpha_f)$ = Slope of Normalised slip angle function for the front tyre
$dC_R/da(\alpha_f)$ = Slope of Normalised slip angle function for the rear tyre
$Fm(L_1)$ = Traction circle radius for the left front (N)
$Fm(L_2)$ = Traction circle radius for the right front (N)
$Fm(L_3)$ = Traction circle radius for the left rear (N)
$Fm(L_4)$ = Traction circle radius for the right rear (N)

The great news is that $Fm(L_1)$ through to $Fm(L_4)$ is given by equation (1) and the k_a and k_b terms for all of these equations are given to you by using ChassisSim and the process we discussed from tyre load modelling from nothing. The tyre loads L_1 to L_4 are given by,

$$\begin{aligned} L_1 &= L_{SF} + \frac{pr \cdot m_t \cdot a_y \cdot h}{tm} \\ L_2 &= L_{SF} + \frac{pr \cdot m_t \cdot a_y \cdot h}{tm} \\ L_3 &= L_{SR} + \frac{(1-pr) \cdot m_t \cdot a_y \cdot h}{tm} \\ L_4 &= L_{SR} + \frac{(1-pr) \cdot m_t \cdot a_y \cdot h}{tm} \end{aligned} \qquad (9)$$

The last bit in this process is deducing calculating the slip angle derivatives. Fortunately there are some techniques that can help us along the way. The first thing to get a handle on what the slopes of the normalised tyre force curve look like. There are a couple of approaches you can use but to get you going but let me get you started on the normalised ChassisSim slip angle curve that has worked very well. This is shown in Table - 2

Table – 2 – Plot of normalised ChassisSim slip angle derivatives

Slip angle (deg)	Slip angle (rad)	$\delta C/d\alpha$
0	0	14.323
1	0.0175	13.925
2	0.0349	12.731
3	0.0524	10.742
4	0.0698	7.9567
5	0.0872	4.375
6	0.1047	0

The last bit in the process is choosing what slip angles to take these calculations from. Looking at Table - 1 you'd be nuts to choose 6 deg. The slopes are zero and it makes no sense. In light of this the procedure will be to set the rear slip angle at 5 deg. Then the front slip angle will be given by,

$$\alpha_F = \frac{b \cdot (Fm(L_3) + Fm(L_4))}{a \cdot (Fm(L_1) + Fm(L_2))} \cdot \alpha_R \qquad (10)$$

Here we have

a = Moment arm of front axle to centre of gravity (m)

b = Moment arm of rear axle to centre of gravity
a_f = Front slip angle
a_r = Rear slip angle

Bear in mind equation (10) isn't something that is set in stone. It is an approximation to help you get an expectation of the relationship between the front and rear slip angles so you can calculate the stability index.

At this point in the game it would be worth giving everyone a reminder about how to calculate the stability index. So we can put some numbers to this let's illustrate via some F3 numbers. This is summarised in Table - 3,

Table – 3 – Typical values for Stability index based around an F3 car

Item	Description	Value
Fm_1+Fm_2	Sum of traction circle radius for the front	5000 N
Fm_1+Fm_2	Sum of traction circle radius for the rear	7000 N
a	Distance of front axle to the c.g	1.6 m
b	Distance of rear axle to the c.g	1.1 m
wb	Wheel base	2.7

Let's say the front slip angle is 5 deg and the rear slip angle is 4 deg. Using equation (8) and the derivatives from Table 2 the stability index is,

$$C_f = \left.\frac{\partial C_f}{\partial \alpha_f}\right|_{\alpha=\alpha_f} \cdot (F_{m1} + F_{m2}) = 4.375 \times 5000 = 21875$$

$$C_r = \left.\frac{\partial C_r}{\partial \alpha_r}\right|_{\alpha=\alpha_r} \cdot (F_{m3} + F_{m4}) = 7.9567 \times 7000 = 55760$$

$$C_T = C_f + C_r = 77634$$

$$stbi = SM / wb \approx \frac{a \cdot C_f - b \cdot C_r}{C_T \cdot wb}$$

$$= \frac{1.6 \times 21875 - 1.1 \times 55760}{77634 \times 2.7}$$

$$= -0.125$$

Let's now reverse the case and consider the oversteer case, where the front slip angle is 4 deg and the rear slip angle is 5 deg. Again evaluating equation 4 we see,

$$C_f = \left.\frac{\partial C_f}{\partial \alpha_f}\right|_{\alpha=\alpha_f} \cdot (F_{m1} + F_{m2}) = 7.9567 \times 5000 = 39783.5$$

$$C_r = \left.\frac{\partial C_r}{\partial \alpha_r}\right|_{\alpha=\alpha_r} \cdot (F_{m3} + F_{m4}) = 4.375 \times 7000 = 30625$$

$$C_T = C_f + C_r = 70408.5$$

$$stbi = SM \,/\, wb \approx \frac{a \cdot C_f - b \cdot C_r}{C_T \cdot wb}$$

$$= \frac{1.6 \times 39783.5 - 1.1 \times 30625}{70408.5 \times 2.7}$$

$$= 0.157$$

Let's now tie all this together into a process of how you find the magic number. The process is summarised below,

- Using equations (5) - (7) plot lateral force vs lateral load transfer.
- The maximum value of this is your start value for lateral load transfer.
- Then using the stability index increase the lateral load transfer to get the desired stability index.
- As a rough rule of thumb go for about -0.025

The great thing about all this is this readily lends itself to an excel sheet. You simply take the lateral acceleration and speed you are interested in. You calculate the front and rear lateral forces and then curve fit the slip angle derivatives. Then using equation (10) and keeping the rear slip angle fixed you can see where the front slip angles are and calculate the stability index accordingly. While this won't be exact it will get you in the ball park.

As a case in point I did this for an F3 type car at a cornering speed of 200 km/h and a lateral acceleration of 1.8g. The results are summarised in Table - 4

Table-4 - Results of Lateral load transfer vs the stability index

Lateral Load transfer	Total lateral force (N)	Projected front slip angle (deg)	Stability index
0.1	21952.64	4.24	0.162
0.2	22264.4	4.42	0.13
0.3	22479.4	4.6	0.09
0.4	22597.6	4.80	0.05
0.5	22619.05	5.01	-0.00291
0.6	22543	5.24	-0.072
0.7	22371	5.51	-0.166
0.8	22102.6	5.8	-0.303
0.9	21736.9	6.14	-0.524

To say these figures are fascinating is an understatement. As we can see the peak lateral force occurs at a front lateral load transfer of 0.5. Not surprisingly the stability index is very marginal at -0.00291. What is interesting is when we go to a lateral load transfer factor of 0.6 we drop only 80N of force but the stability index drops to

-0.072. This is a big change in handling. What is even more interesting though is the spread of forces is only about 1000N or about 4%. However we see large fluctuations of the stability index. What this shows is the magic number isn't just about maximum grip but it's about dialling in the handling of the car that you want.

To finish off this discussion it would be worth discussing the effect of total lateral acceleration on lateral load transfer number we want. For absolute grip it won't have an impact, but for stability it will. This can be quantified by the following,

$$\begin{aligned} F_{yf} &= 2 \cdot TC_{RAD}(L_{SF}) - 2 \cdot k_{af} \cdot k_{bf} \cdot \left(\frac{pr \cdot m_t \cdot a_y \cdot h}{tm} \right)^2 \\ F_{yr} &= 2 \cdot TC_{RAD}(L_{SR}) - 2 \cdot k_{ar} \cdot k_{br} \cdot \left(\frac{(1-pr) \cdot m_t \cdot a_y \cdot h}{tm} \right) \end{aligned} \qquad (11)$$

This in turn impacts on the stability index we discussed. Consequently when incorporating this in an excel sheet you need to be entering the lateral acceleration and corner speeds you are interested in.

However everything we have discussed here is all for nothing if you have not done your simulation modelling. This is where ChassisSim and it's tyre modelling utilities are worth their weight in gold. The reason they are worth their weight in gold is you can use these tools to find out the k_a and k_b for the front and rear tyres from race data. Without these you are just guessing. Once you have this, filling in the details of all of the above becomes a fait accompli which is one of the key reasons any race engineer worth their salt would be crazy not too have a simulation tool such as ChassisSim in their back pocket.

In closing we have discussed in detail the origin of the magic number but more importantly how to calculate it. You start from the maximum grip number and then you add front roll distribution till you satisfy your stability requirements. Remember the final number will always be a trade off between maximum grip and drivability as shown clearly in Table 3. While this won't give you the perfect setup straight out of the box it will get you to a very good start point. However without the appropriate simulation package such as ChassisSim to fill in these numbers this will all be nothing. If you combine these analytical tools it gives you a very powerful tool to truly understand what your car is doing.